# Implementation with Statistics

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**Abstract**

A method of implementation is introduced for collective decision problems when only some statistics about the type space $\Omega$ are known: First, use those statistics to whittle $\Omega$ down to a high probability event $\Omega^*$. Then, design a mechanism $M^*$ to ex-post implement the desired outcome with $\Omega^*$ as the type space. Viewed as a mechanism over the true type space $\Omega$, $M^*$ is typically not ex-post. However, under a weaker solution concept I call *$\varepsilon$-ex-post equilibrium*, $M^*$ implements the desired outcome in a high probability subevent of $\Omega^*$. An application to a dynamic allocation problem shows how implementation with statistics can yield significantly better results than ex-post implementation.

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# 1 Introduction

In a typical collective decision problem involving a mechanism designer and a set of agents, it is unlikely that the true probability measure governing the type space $\Omega$ will be common knowledge, or even known to anyone. On the other hand, having common knowledge of some basic statistics about $\Omega$ seems quite plausible. Such knowledge could, for example, be extracted from data generated by previous related decision problems or interactions with similar players.

In such a scenario, without a prior, Bayesian implementation is not feasible. Designing a mechanism that ex-post implements the desired outcome is an option, but that would involve ignoring the known statistics about $\Omega$, which, intuitively, could be quite suboptimal: For example, suppose $\Omega$ is the set of positive reals, and it is known that the expected value of the type is 1. This single statistic implies that there is a less than 1% chance the realized type exceeds 100. Or, suppose it is known that the type involves many independent draws from a distribution – a common situation in dynamic decision problems. Then a Law of Large Numbers style argument implies that the realized type is highly likely to lie in a tiny sliver of the type space.

In this paper, I present an approach to implementation, related to the ex-post approach, that can incorporate those potentially important statistics: The mechanism designer whittles the true type space $\Omega$ down to an event, $\Omega^*$, that the common knowledge statistics imply is of high probability. The mechanism designer then designs a direct mechanism $M^*$ to ex-post implement the desired outcome treating $\Omega^*$ as the type space. Compose $M^*$ with a retraction mapping $[\cdot] : \Omega \to \Omega^*$ to get a direct mechanism $M^* \circ [\cdot]$ over the true type space $\Omega$, and call it a *statistical mechanism.* The mechanism designer then uses such a statistical mechanism on the agents.

In what sense does using a statistical mechanism "work?" In the paper, I define what it means for truth-telling to be an $\varepsilon$*-ex-post equilibrium* of a statistical mechanism. $\varepsilon$-ex-post equilibrium is a slight weakening of ex-post equilibrium. When truth-telling is an $\varepsilon$-ex-post equilibrium, I argue that one can expect, with high probability, all agents to report the truth at all dates. I then show that, if it is known that $\Omega^*$ is of sufficiently high probability, then truth-telling is an $\varepsilon$-ex-post equilibrium of $M^* \circ [\cdot]$, and, consequently, the mechanism designer can expect with high probability that all agents report the truth at all dates when faced with $M^* \circ [\cdot]$. Moreover, recall, $M^*$ ex-post implements the desired outcome over $\Omega^*$. Putting these two facts together leads us to conclude that, if it is known that $\Omega^*$ is of sufficiently high probability, then $M^* \circ [\cdot]$ implements the desired outcome with high probability.

If the mechanism designer is comfortable with implementation on a high probability event of the type space rather than implementation over the entire type space, then this statistical approach to implementation can yield significantly "better" solutions to the collective decision problem than ex-post implementation. In particular, in settings where there is a notion of cost, it can yield significantly cheaper mechanisms.

In the second half of the paper, I demonstrate this by comparing the two approaches in the context of a repeated resource allocation problem. The setting is quasilinear, agents have private values and are protected by limited liability, and the mechanism designer can make nonnegative transfers to the agents in an effort to implement the efficient allocation of resources each date. I show that when the number of agents and dates goes to infinity, the cheapest efficient ex-post mechanism – which is essentially just a sequence of VCG mechanisms – has an infinite cost-to-surplus ratio. On the other hand, suppose agents are patient and some "Law of Large Numbers style" statistics are known that still provide enough room for significant departures from the baseline iid case. Then, in the limit, the mechanism designer can – via a statistical mechanism I call the *linked VCG mechanism* – implement the efficient allocation almost surely at a cost-to-surplus ratio of zero.

The concept of $\varepsilon$-ex-post equilibrium is related to notions of approximate strategy-proofness recently developed by Lee (2017) and Azevedo and Budish (2019). Also related are the contemporaneous perfect $\varepsilon$-equilibrium of Mailath, Postlewaite, and Samuelson (2005) and the dynamic ex-post implementation concept of Bergemann and Välimäki (2002). On the other hand, implementation with statistics is quite distinct from virtual implementation even though both methods involve the idea of implementation with high probability. See, for example, Abreu and Matsushima (1992). In virtual implementation, the high probability requirement is imposed at the ex-post rather than ex-ante stage – that is, for each type profile, virtual implementation demands that the mechanism implements the desired outcome with high probability. Thus, it is best to view virtual implementation and implementation with statistics as two orthogonal departures from ex-post implementation. In principle, one could even combine the two methods of implementation (although this is not explored in the current paper): First, generate an $\Omega^*$ as in implementation with statistics, then virtually implement the desired outcome with $\Omega^*$ as the type space.

In the application to repeated resource allocation, my work on linking VCG mechanisms is related the work of Holmström (1979) on VCG mechanisms over restricted preference domains. See also Green and Laffont (1977). In the many agents and dates limit, the linked VCG mechanism that implements the efficient allocation almost surely at a cost-to-surplus ratio of zero resembles a budget mechanism. Thus, my work reveals a surprising connection between VCG mechanisms and budget mechanisms. A number of papers have shown how budget mechanisms can align incentives across multiple decision problems when transfers are unavailable. The linking mechanism of Jackson and Sonnenschein (2007) is one such budget mechanism (see, also, Frankel 2014), and it is explicitly designed for repeated decision problems like the resource allocation one considered here. The linking mechanism works in the large numbers limit when the decision problems across dates are iid with known distribution, implementing the efficient allocation almost surely for free. In contrast, the linked VCG mechanism is not free, but, as I will show, it can be designed to work about as cheaply as possible across a wide range of statistical settings, including those that are far from the iid large numbers limit.

# 2 Model

## 2.1 Decision Problems

Given integers $N \geq 2$ and $T \geq 1$, an $N$-agent $T$-date *decision problem* is a triple $(\Omega, D, U)$. $\Omega = \Pi_{1\leq n\leq N,\ 1\leq t\leq T}\Omega_t^n$ is the type space, where each $\Omega_t^n$ is a finite set of date $t$ types for agent $n$. $D = \Pi_{t=1}^T D_t$ is a finite set of decision sequences. $U^n : D \times \Omega^n \to \mathbb{R}$ is agent $n$'s private values payoff function and is defined to be

$$U^n(d, \omega^n) = \sum_{t=1}^{T} \beta^{t-1} u_t^n(d|_t, \omega^n|_t),$$

where $\beta \in (0, 1]$ is the discount factor and $u_t^n : D|_t \times \Omega^n|_t \to \mathbb{R}$ is agent $n$'s date $t$ utility function, which depends on the history of decisions, $d|_t$, and agent $n$ types, $\omega^n|_t$, up through date $t$.

A *credal set* is a nonempty set of probabilities, $\mathcal{P}$, over $\Omega$. It is common knowledge that $\omega$ is governed by some true probability, call it $P$, lying in $\mathcal{P}$. In a typical application, the credal set will be an infinite set of probabilities that satisfy some commonly known statistics. The only restriction I impose on credal sets is that they consist only of *private* probabilities. A probability $\hat{P}$ is private if $\hat{P}(A^n \times \Omega^{-n}|\ \omega|_t) = \hat{P}^n(A^n|\ \omega^n|_t)$ for all $n$ and $A^n \subset \Omega^n$. This restriction implies it is common knowledge that, at each date $t$, the distribution of agent $n$'s future types is independent of other agents' type histories conditional on agent $n$'s type history.

Lastly, I assume it is common knowledge each agent $n$ knows his own marginal, $P^n$, of the true probability.

## 2.2 Notation

For an object $\cdot_t^n$ indexed by agents and dates, let the superscript denote the agent index and the subscript denote the date index. Let $\cdot^n$ denote agent $n$'s *sequence* of $\cdot_t^n$ across all dates and let $\cdot_t$ denote the date $t$ *profile* of $\cdot_t^n$ across all agents. Let $\cdot$ denote the array of $\cdot_t^n$ across agents and dates. If an object $\cdot^n$ is only indexed by agents, then let $\cdot$ denote the profile of $\cdot^n$ across agents. If an object $\cdot_t$ is only indexed by date, then let $\cdot$ denote the sequence of $\cdot_t$ across all dates, and let $\cdot|_t$ denote the subsequence of $\cdot$ up through date $t$.

Let $A$ and $B$ be two sets of sequences. A map $f : A \to B$ is adapted is $a|_t = a'|_t \Rightarrow f(a)|_t = f(a')|_t$.

## 2.3 Statistical Mechanisms

Given subsets $\Omega^{*n} \subset \Omega^n$ for each agent $n$, define the event $\Omega^* = \Pi_{1\leq n\leq N}\Omega^{*n}$. A *retraction* $[\cdot]^n$ is an adapted map from $\Omega^n$ to $\Omega^{*n}$ that is the identity on $\Omega^{*n}$.

**Lemma 1.** *There exists a retraction from $\Omega^n$ to $\Omega^{*n}$.*

A direct mechanism over $\Omega^*$ is an adapted map $M^* : \Omega^* \to D$. An ex-post direct mechanism over $\Omega^*$ is an adapted map $M^* : \Omega^* \to D$ satisfying

$$U^n(M^*(\omega^{-n}, \omega^n), \omega^n) \geq U^n(M^*(\omega^{-n}, \hat{\omega}^n), \omega^n) \quad \forall n,\ \omega^{-n} \in \Omega^{*-n},\ \omega^n, \hat{\omega}^n \in \Omega^{*n}.$$

A *statistical mechanism* is an ex-post direct mechanism $M^*$ over some $\Omega^*$ composed with a retraction profile $[\cdot] : \Omega \to \Omega^*$. It is a direct mechanism over $\Omega$.

Given a direct mechanism over $\Omega$, an agent $n$ strategy, $\sigma^n$, consists of a sequence of maps $\sigma_t^n : D|_{t-1} \times \Omega^n|_t \to \Omega_t^n$. Let $\Sigma^n$ denote the set of all agent $n$ strategies. A profile of strategies, $\sigma$, can be viewed as an adapted map $\sigma : \Omega \to \Omega$. Let $id$ be the strategy profile in which all agents report the truth at all dates.

## 2.4 $\varepsilon$-Ex-Post Equilibrium

For the rest of the paper, fix an $\varepsilon > 0$, to be interpreted as "small." In this section, I define what it means for $id$ to be an $\varepsilon$-ex-post equilibrium of a statistical mechanism.

Fix a statistical mechanism, $M^* \circ [\cdot]$, corresponding to some $\Omega^*$. For each agent $n$, date $t$, and type history $\omega^n|_t$ define $R_t^n(\omega^n|_t) :=$

$$\max_{\substack{\tilde{\omega} \in \Omega,\ \hat{\omega}^n \in \Omega^n \\ s.t.\ \tilde{\omega}^n|_t = \omega^n|_t,\ \hat{\omega}^n|_{t-1} = \omega^n|_{t-1}}} \sum_{s=t}^{T} \beta^{s-t} \left[ u_s^n(M^* \circ [\hat{\omega}^n, \tilde{\omega}^{-n}]|_s, \tilde{\omega}^n|_s) - u_s^n(M^* \circ [\tilde{\omega}]|_s, \tilde{\omega}^n|_s) \right].$$

$R_t^n(\omega^n|_t)$ is agent $n$'s maximum regret standing at date $t$ from continuing to play $id^n$ assuming he has played $id^n$ up through date $t-1$ and all other agents play $id^{-n}$.

**Definition.** *An agent $n$ strategy $\sigma^n$ is reasonable against $id^{-n}$ if, for all dates $t$,*

$$P^n(\omega^n \notin \Omega^{*n} \mid \omega^n|_s) \cdot R_s^n(\omega^n|_s) < \varepsilon \quad \forall s \leq t \Rightarrow \sigma_t^n(d|_{t-1}, \omega^n|_t) = \omega_t^n. \tag{1}$$

Consider agent $n$ deciding what to report at date 1 against $id^{-n}$. Because of how a statistical mechanism is constructed, he knows that reporting the truth starting from today is ex-post optimal if $\omega^n \in \Omega^{*n}$. If $\omega^n \notin \Omega^{*n}$, the maximum regret he will experience from reporting the truth starting from today is $R_1^n(\omega^n|_1)$. Thus, if the left side of (1) is satisfied for $t = 1$, then it could be said that, from the perspective of agent $n$ at date 1, reporting the truth starting from today is within $\varepsilon$ of being ex-post optimal. In this case, I assume agent $n$ reports the truth at date 1. The definition is now justified by induction.

One thing worth commenting on about the left hand side of (1): When $t > 1$, agent $n$ observes a nontrivial decision history $d|_{t-1}$, which is informative of other agents' type histories, $\omega^{-n}|_{t-1}$. Since $\omega^{-n}|_{t-1}$ can be informative of $\omega^n$, agent $n$ should, in principle, also condition on $d|_{t-1}$ when formulating his conditional belief about $\omega^n \notin \Omega^{*n}$. This is problematic since, in general, agent $n$ need not know $P$. Fortunately, the fact that agent $n$ knows $P$ is a private probability and observes his own type history implies that his conditional belief about $\omega^n \notin \Omega^{*n}$ is independent of the other agents' type histories, and, therefore, $d|_{t-1}$. Alternatively, we could just assume that agent $n$ is unable to infer anything from observing $d|_{t-1}$, perhaps because it is too mentally taxing to make inferences. In this case, we can drop the private probabilities assumption about credal sets.

Let $\Sigma^n(id^{-n})$ denote the set of all $\sigma^n$ that are reasonable against $id^{-n}$, and let $\Sigma(id)$ denote the set of all profiles of such $\sigma^n$.

**Definition.** *$id$ is an $\varepsilon$-ex-post equilibrium if it is common knowledge that every reasonable strategy profile, $\sigma$, satisfies $P^{-n}(\sigma^{-n}(\omega^{-n}) \neq \omega^{-n}) \leq \varepsilon$ for all $n$.*

$\varepsilon$-ex-post equilibrium generalizes ex-post equilibrium in the following sense: If $id$ is an ex-post equilibrium of a direct mechanism $M$ then it is an $\varepsilon$-ex-post equilibrium of $M$ viewed as a statistical mechanism corresponding to $\Omega^* = \Omega$. Indeed, suppose $id$ is an ex-post equilibrium of $M$. Viewing $M$ as a statistical mechanism corresponding to $\Omega^* = \Omega$, we have $R^n_t(\omega^n|_t) = 0$. The definition of reasonability now implies $\Sigma(id) = \{id\}$. When $\Sigma(id) = \{id\}$, it is obviously common knowledge that every reasonable strategy profile, $\sigma$, satisfies $P^{-n}(\sigma^{-n}(\omega^{-n}) \neq \omega^{-n}) \leq \varepsilon$ for all $n$. Therefore, $id$ is an $\varepsilon$-ex-post equilibrium.

It is also worth comparing the definition of $\varepsilon$-ex-post equilibrium with that of perfect Bayesian equilibrium. In a perfect Bayesian equilibrium,

I. Each agent's beliefs are "reasonable" given their own strategies and their conjectures about the other agents' strategies (Bayes' Rule).

II. Each agent's strategies are "reasonable" given their beliefs and their conjectures about the other agents' strategies (sequential rationality).

III. Each agent correctly conjectures the other agents' strategies.

In terms of $\varepsilon$-ex-post equilibrium, the relevant "beliefs" an agent must form are those about the probability his type will land in $\Omega^{*n}$. Since each agent knows his own marginal, $P^n$, these beliefs are $\{P^n(\omega^n \notin \Omega^{*n} \mid \omega^n|_t)\}_{t=1}^T$ which, by definition, satisfy Bayes' Rule. The definition of what it means for $\sigma^n$ to be reasonable corresponds to the sequential rationality condition. Finally, the condition that it is common knowledge that, in any reasonable strategy profile, the other agents' strategies must be $\varepsilon$-close to truth-telling corresponds to (a slight weakening of) the correct conjectures condition.

When $id$ is an $\varepsilon$-ex-post equilibrium of a statistical mechanism, I will interpret it to mean that one can expect some reasonable strategy profile $\sigma \in \Sigma(id)$ to be played. In a typical application, the set, $\Sigma(id)$, will not be known to anyone. Nevertheless, one will still be able to deduce things just from the knowledge that the strategy profile being played belongs in $\Sigma(id)$. In particular, one knows that the probability all agents will report the truth at all dates is at least $1 - 2\varepsilon$.

### 2.5 Implementation with Statistics

**Proposition 1.** *Let $M^* \circ [\cdot]$ be a statistical mechanism corresponding to some $\Omega^*$. Let $\overline{R} \geq \varepsilon$ be an upper bound of $R_t^n(\omega^n|_t)$ for all $n, t, \omega^n|_t$. For any $c \leq \varepsilon$, if it is common knowledge that $P^n(\omega^n \notin \Omega^{*n}) \leq \frac{c}{N-1} \cdot \frac{\varepsilon}{\overline{R}}$ for all agents $n$, then id is an $\varepsilon$-ex-post equilibrium of $M^* \circ [\cdot]$.*

A *desired outcome* is a correspondence $DO : \Omega \to D$ where $DO(\omega)$ is the set of decisions in which the mechanism designer wants the implemented decision to be. For example, in an auction decision problem, a decision would be an allocation of the object along with payments from the bidders, and, for a mechanism designer who desires efficiency, $DO(\omega)$ could be the set of decisions that involve allocating the object to the bidder with the highest valuation.

Let $M^*$ be an ex-post direct mechanism over $\Omega^*$. We say $M^*$ ex-post implements the desired outcome over $\Omega^*$ if $M^*(\omega) \in DO(\omega)$ for all $\omega \in \Omega^*$. Let $M^* \circ [\cdot]$ be a statistical mechanism. We say $M^* \circ [\cdot]$ implements the desired outcome with probability at least $p$ if $id$ is an $\varepsilon$-ex-post equilibrium and it is common knowledge that $P(M^* \circ [\sigma(\omega)] \in DO(\omega)) \geq p$ for all $\sigma \in \Sigma(id)$.

We are now ready to state the central result of implementation with statistics:

**Corollary 1.** *Suppose $M^*$ ex-post implements the desired outcome over some $\Omega^*$. Let $c \leq \varepsilon$ and $\overline{R} \geq \varepsilon$ be an upper bound of $R_t^n(\omega^n|_t)$ for all $n, t, \omega^n|_t$. If it is common knowledge $P^n(\omega^n \notin \Omega^{*n}) \leq \frac{c}{N-1} \cdot \frac{\varepsilon}{\overline{R}}$ for all agents $n$, then any corresponding statistical mechanism implements the desired outcome with probability at least $1 - \frac{Nc}{N-1}$.*

## 3 Application: Repeated Resource Allocation

### 3.1 A Model of Repeated Resource Allocation

A principal (she) possesses a quantity $\overline{q}$ of a divisible, durable resource. She repeatedly allocates this resource to a set of $N \geq 2$ agents across $T \geq 1$ dates.

At each date $t$, each agent $n$ is endowed with $\omega_t^n \in [0, \infty)$ units of a project type, $f$. $f$ is a strictly concave, $C^1$ function $f : [0, \infty) \to [0, \infty)$ that maps resource quantity to payoff. Assume $f'(0) < \infty$.

An allocation array $a$ assigns agent $n$ at date $t$ an amount $a_t^n \geq 0$ of the resource, subject to feasibility constraints, $\sum_{n=1}^N a_t^n \leq \overline{q}$ for all $t$. A transfer profile $w$ specifies a profile of nonnegative payments from the principal to the agents at date $T$.

The principal desires to efficiently allocate her resource each date.

One application of this model is to an organization's problem of designing an *internal talent marketplace*. Instead of having a static collection of employee-job matchings, many organizations are reimagining work as a flow of discrete tasks that need to be assigned to available employees through some dynamic mechanism. See Smet, Lund and Schaninger (2016).

This problem can be viewed through the repeated resource allocation model: The principal corresponds to the organization's headquarters and the agents correspond to various departments. Projects are departmental tasks. The stock of durable resources is the organization's pool of employees parameterized by hours of labor per date, where a date could be, say, one month. Transfers from the principal to agents correspond to incentive pay for department managers.

## 3.2 The Induced Decision Problem

The repeated resource allocation model defines an $N$-agent $T$-date decision problem:

- $\Omega = [0, \infty)^{NT}$,
- $D = \{(a, w) \mid \sum_{n=1}^{N} a_t^n \leq \overline{q} \;\; \forall t \;\text{ and }\; w^n \geq 0 \;\; \forall n\}$, and
- $U^n((a, w), \omega) = U^n((a^n, w^n), \omega^n) = \sum_{t=1}^{T} \beta^{t-1} \omega_t^n f\left(\frac{a_t^n}{\omega_t^n}\right) + \beta^{T-1} w^n$ for all $n$.

In addition, define the following auxiliary quantities,

- agent $n$ surplus: $S^n((a, w), \omega) = S^n(a^n, \omega^n) = \sum_{t=1}^{T} \beta^{t-1} \omega_t^n f\left(\frac{a_t^n}{\omega_t^n}\right)$,
- total surplus: $S((a, w), \omega) = S(a, \omega) = \sum_{n=1}^{N} S^n(a^n, \omega^n)$, and
- cost: $C((a, w), \omega) = C(w) = \sum w^n$.

A direct mechanism can be expressed as a pair of adapted maps $(A, W) : \Omega \to D$ consisting of an allocation map and a transfer map. The efficient allocation map is the unique allocation map, $\boldsymbol{A}$, satisfying

$$\boldsymbol{A}_t^n(\omega) = \frac{\omega_t^n}{\sum_{m=1}^{N} \omega_t^m} \cdot \overline{q} \qquad \forall \omega \in \Omega.$$

A direct mechanism $(A, W)$ is efficient if $A \equiv \boldsymbol{A}$.

The principal's desire to efficiently allocate her resource each date induces an implementation problem where the desired outcome is $DO(\omega) = \{(\boldsymbol{A}(\omega), w) \mid w^n \geq 0 \; \forall n\}$ for all $\omega \in \Omega$.

I now compare the ex-post and stylized approaches to implementation, with a focus on which approach is cheaper for the principal.

## 3.3 The Unlinked VCG Mechanism

Suppose the principal wants to ex-post implement the efficient allocation. One option is to run a separate Vickrey-Clark-Groves (VCG) mechanism each date, paying each agent the sum of all other agents' contributions to surplus:

**Definition.** *The unlinked VCG mechanism $(\boldsymbol{A}, V)$ is the efficient direct mechanism with transfer map defined as follows: For all $\omega \in \Omega$,*

$$V^n(\omega) = \sum_{m \neq n} \sum_{t=1}^{T} \beta^{t-T} \omega_t^m f\left(\frac{\boldsymbol{A}_t^m(\omega)}{\omega_t^m}\right).$$

**Proposition 2.** *The unlinked VCG mechanism is the cheapest efficient ex-post direct mechanism: Let $(\boldsymbol{A}, W)$ be any efficient ex-post direct mechanism. Then for every $\omega \in \Omega$, we have $C(V(\omega)) \leq C(W(\omega))$.*

Proposition 2 is a corollary of Proposition 3 below.

Even though the unlinked VCG mechanism is the cheapest mechanism that ex-post implements the efficient allocation, it is still expensive with cost-to-surplus ratio

$$\frac{C(\omega)}{S(\boldsymbol{A}(\omega), \omega)} = N - 1.$$

In particular, as the number of agents tends to infinity, so does the cost-to-surplus ratio.

## 3.4 The Linked VCG Mechanism

**Definition.** *Given $\Omega^*$, the linked VCG mechanism $(\boldsymbol{A}|_{\Omega^*}, V^*)$ is the efficient direct mechanism over $\Omega^*$ with transfer map defined as follows: For all $\omega \in \Omega^*$,*

$$V^{*n}(\omega) = V^n(\omega) - \underset{\hat{\omega}^n \in \Omega^{*n}}{\arg\min}\, V^n(\omega^{-n}, \hat{\omega}^n).$$

The linked VCG mechanism $(\boldsymbol{A}|_{\Omega^*}, V^*)$ is obviously an efficient ex-post direct mechanism over $\Omega^*$. In fact,

**Proposition 3.** *If $\Omega^*$ is smoothly path-connected, then the linked VCG mechanism $(\boldsymbol{A}|_{\Omega^*}, V^*)$ is the cheapest efficient ex-post direct mechanism over $\Omega^*$.*

Proposition 3 is a consequence of Theorem 1 of Hölmström (1979) about the necessity of VCG mechanisms over restricted domains. The proof is a straightforward application of the envelope theorem. Since the main theorem we are about to prove, Theorem 1, does not rely on Proposition 3, its proof is omitted.

Proposition 1 implies that any statistical mechanism corresponding to $(\boldsymbol{A}|_{\Omega^*}, V^*)$ implements the efficient allocation with high probability provided it is common knowledge $P^n(\Omega^{*n})$ is sufficiently close to 1 for each agent $n$. As an abuse of nomenclature, call any such statistical mechanism a linked VCG mechanism as well, and from now on I will denote it by $(\Omega^*, V^*)$.

I now show, as the number of agents and dates goes to infinity, assuming agents are patient and $\mathcal{P}$ implies common knowledge of some "Law of Large Numbers style" statistics, then, by taking the statistical approach to implementation, the principal can implement the efficient allocation almost surely at a cost-to-surplus ratio of zero. This is in stark contrast to taking the ex-post approach, which would entail a cost-to-surplus ratio of infinity.

## 3.5 A Family of Repeated Resource Allocation Models

Fix a quantity $q > 0$ of the resource and a project type $f$. Consider the family of decision problems parameterized by $N$ satisfying $(\overline{q}(N), f(N)) = (Nq, f)$, $T(N) = N$ and $\beta(N) = 1$. Refer to the member of the family with $N$-agents as the $N$-agent decision problem. Throughout the analysis below, we may append $(N)$ to a parameter to emphasize that it is belongs to the $N$-agent decision problem.

Assume the family of credal sets $\{\mathcal{P}(N)\}_{N\geq 2}$ satisfies the following:

**Assumption 1.** *There exist*

- *a bounded set of positive reals* $\{\omega^{avg}\} \cup \{\omega^{avg,n}\}_{n=1}^{\infty}$,
- *weakly decreasing functions* $g, G : [0,\infty) \to [0,\infty)$ *satisfying* $\lim_{z\to\infty} z^5 g(z) = \lim_{z\to\infty} z^3 G(z) = 0$,
- *an increasing function* $I : (0,\infty) \to (0,\infty)$ *satisfying* $\lim_{x\to\infty} I(x) = \infty$,

*such that, for each $N$-agent decision problem, it is common knowledge that*

$$P(N)\left[\left|\frac{\sum_{s\neq t}\omega_s^n}{N-1} - \omega^{avg,n}\right| > x\right] \leq g(I(x)(N-1)) \qquad \forall x > 0,\ n,t \leq N,$$
$$P(N)\left[\left|\frac{\sum_{m\neq n}\omega_t^m}{N-1} - \omega^{avg}\right| > x\right] \leq G(I(x)(N-1)) \qquad \forall x > 0,\ t,n \leq N.$$

If it is common knowledge that the array of endowments $\{\omega_t^n\}_{1\leq n,t<\infty}$ is iid with mean $\mu$ and variance $\sigma^2$, then the Central Limit Theorem implies Assumption 1 is satisfied. However, the concentration inequalities of Bernstein and Hoeffding imply that there is room for significant departures from the baseline iid case while still satisfying Assumption 1. For example, suppose there exist positive values $\mu$ and $\overline{\omega}$ such that it is common knowledge that the array of endowments $\{\omega_t^n\}_{1\leq n,t<\infty}$ is independent but not necessarily identically distributed, with mean $\mu$ and upper bound $\overline{\omega}$. Then Hoeffding's Inequality says that, for each $N$,

$$P(N)\left[\left|\frac{\sum_{s\neq t}\omega_s^n}{N-1} - \mu\right| > x\right] \leq 2\exp\left(-\frac{2(N-1)x^2}{\overline{w}^2}\right),$$
$$P(N)\left[\left|\frac{\sum_{m\neq n}\omega_t^m}{N-1} - \mu\right| > x\right] \leq 2\exp\left(-\frac{2(N-1)x^2}{\overline{w}^2}\right),$$

for all $x > 0$ and $t, n \leq N$. By defining $g(z) = G(z) = 2\exp(-z)$ and $I(x) = \frac{2x^2}{\overline{\omega}^2}$, we see that Hoeffding's Inequality implies Assumption 1. In addition, it is clear that Assumption 1 allows for the possibility of significant correlation between endowments that differ in both the agent and time dimensions. This means many pairs of endowments can be highly correlated. For example, let $s_2, s_3 \ldots$ be a sequence of iid positive random variables with mean $\mu$ and variance $\sigma^2$, and let $\omega_t^n = s_{n+t}$. Then Assumption 1 is satisfied while any pair of endowments with the same sum of time and agent indices are perfectly correlated.

**Theorem 1.** *There exists a family, $\{(\Omega^*(N), V^*(N))\}_{N\geq 2}$, of linked VCG mechanisms, one for each $N$-agent decision problem, such that id is an $\varepsilon$-ex-post equilibrium of each mechanism, and it is common knowledge that*

$$\lim_{N\to\infty} \inf_{\sigma\in\Sigma(N)(id)} P(N)\left(\omega \in \Omega^*(N),\ \sigma(\omega) = \omega\right) = 1,$$

$$\lim_{N\to\infty} \inf_{\sigma\in\Sigma(N)(id)} \frac{\mathbf{E}_{P(N)} S(\boldsymbol{A}|_{\Omega^*(N)} \circ [\sigma(\omega)], \omega)}{\mathbf{E}_{P(N)} S(\boldsymbol{A}(\omega), \omega)} = 1,$$

*and*

$$\lim_{N\to\infty} \sup_{\sigma\in\Sigma(N)(id)} \frac{\mathbf{E}_{P(N)} C(V^* \circ [\sigma(\omega)])}{\mathbf{E}_{P(N)} S(\boldsymbol{A}|_{\Omega^*(N)} \circ [\sigma(\omega)], \omega)} = 0.$$

Theorem 1 says that in the limit it is possible to design a linked VCG mechanism that will almost surely induce all agents to report the truth at all dates and allocate resources efficiently, at an expected cost to expect surplus ratio of 0.

Let us gain some intuition for the result. When $N$ is large, the efficient allocation of resources to each date $t$ project is

$$\frac{Nq}{\sum_{m=1}^{N} \omega_t^m} \approx \frac{q}{\omega^{avg}}.$$

Thus, the efficient marginal productivity of resource is approximately always $f' := f'(q/\omega^{avg})$. When $N$ is large, we can whittle down each agent's type space to be approximately

$$\Omega^{*n}(N) \approx \left\{ \omega^n \;\middle|\; \frac{\sum_{t=1}^{N} \omega_t^n}{N} = \omega^{avg,n} \right\}$$

while still ensuring that the probability that $\omega \in \Omega^*(N)$ and $\sigma(\omega) = \omega$ under any reasonable $\sigma$ is high. Fix such an $\omega$. Let us now approximate agent $n$'s transfer under the linked VCG mechanism $(\Omega^*(N), V^*(N))$.

$$\begin{aligned} V^{*n}(N)(\omega) &= V^n(N)(\omega) - \underset{\hat{\omega}^n \in \Omega^{*n}(N)}{\arg\min}\; V^n(N)(\omega^{-n}, \hat{\omega}^n) \\ &= \underset{\hat{\omega}^n \in \Omega^{*n}(N)}{\arg\max}\, (V^n(N)(\omega^{-n}, 0) - V^n(N)(\omega^{-n}, \hat{\omega}^n)) \\ &\qquad - (V^n(N)(\omega^{-n}, 0) - V^n(N)(\omega)) \\ &\approx \underset{\hat{\omega}^n \in \Omega^{*n}(N)}{\arg\max} \left( \sum_{t=1}^{N} \hat{\omega}_t^n \right) \cdot \frac{q}{\omega^{avg}} \cdot f' - \left( \sum_{t=1}^{N} \omega_t^n \right) \cdot \frac{q}{\omega^{avg}} \cdot f' \\ &\approx N\omega^{avg,n} \cdot \frac{q}{\omega^{avg}} \cdot f' - \left( \sum_{t=1}^{N} \omega_t^n \right) \cdot \frac{q}{\omega^{avg}} \cdot f' \qquad (2) \\ &\approx 0 \text{ fraction of } N. \end{aligned}$$

Summing over all agents yields a cost that is approximately a zero fraction of $N^2$. Since expected surplus is on the order of $N^2$, the expected cost to expected surplus ratio is approximately zero.

As equation (2) in the derivation of agent $n$'s approximate transfer makes clear, in the limit, the linked VCG mechanism resembles a budget mechanism where each agent $n$ is given a budget of $N\omega^{avg,n} \cdot \frac{q}{\omega^{avg}} \cdot f'$ and the price of the resource is set to $f'$ each date. Each agent is then free to choose how much resources to buy each date subject to his budget constraint. Each agent optimally purchases an amount $\frac{q}{\omega^{avg}}$ of the resource for each of his projects – which equates marginal product with marginal cost. The assumption about each agent's average endowment implies that this purchasing strategy is approximately budget balanced.

Given the connection to budget mechanisms, it is natural to wonder how the linked VCG mechanism compares with the linking mechanism of Jackson and Sonnenschein (2007) (hereafter J-S), which is an explicit budget mechanism designed for repeated decision problems like the one considered here. In the context of the repeated resource allocation problem, the linking mechanism of J-S allows each agent $n$ to report whatever endowment process he wants subject to the budget constraint that the empirical distribution of endowments matches the probability distribution of endowments.

When there are many dates, and $\{\omega_t^n\}$ is independent across agents and dates, and identically distributed with known distribution across dates holding the agent fixed, the linking mechanism implements the efficient allocation almost surely for free. In contrast, the best the linked VCG mechanism can do under these probabilistic assumptions is to implement the efficient allocation almost surely at an expect cost that is a vanishingly small fraction of expected surplus – and even this result requires there to be many agents in addition to many dates.

However, the linked VCG mechanism does have some strengths. Our discussion following Assumption 1 implies that Theorem 1 remains valid in settings that significantly relax the probabilistic assumptions of J-S: Many pairs of endowments can be highly correlated; holding the agent fixed, endowments across dates can be far from identically distributed; and the mechanism designer need not know any agent's endowment distribution.

Perhaps the biggest strength of the linked VCG mechanism is that it is not just a single mechanism, designed to work perfectly in the large numbers limit. Given any credal set, the mechanism designer can try her best to whittle $\Omega$ down to a sufficiently high probability $\Omega^*$ and create the corresponding linked VCG mechanism. This linked VCG mechanism will implement the efficient allocation with high probability. Moreover, if $\Omega^*$ is smoothly path-connected, then it is optimal treating $\Omega^*$ as the type space. A reasonable interpretation of these facts is that the linked VCG mechanism works well – if not perfectly – in a wide variety of statistical settings.

# 4 Appendix

*Proof of Lemma 1.* Let $t(\omega^n)$ be the first date $t$ for which there does not exist an $\hat{\omega}^n \in \Omega^{*n}$ such that $\hat{\omega}^n|_t = \omega^n|_t$. $t(\omega^n)$ is a stopping time. If $\omega^n \in \Omega^{*n}$, then set $t(\omega^n) = T+1$. For each $\omega^n|_{t(\omega^n)-1}$, select a $\hat{\omega}^n \in \Omega^{*n}$ such that $\hat{\omega}^n|_{t(\omega^n)-1} = \omega^n|_{t(\omega^n)-1}$. Define $[\omega^n]$ to be the $\hat{\omega}^n$ selected given $\omega^n|_{t(\omega^n)-1}$. It is clearly the identity function over $\Omega^{*n}$.

To verify $[\cdot]^n$ is adapted, let $\omega'^n, \omega''^n \in \Omega^n$ satisfy $\omega'^n|_t = \omega''^n|_t$ for some $t$. Since $t(\omega)$ is a stopping time, it must be that either $t \geq t(\omega'^n) = t(\omega''^n)$ or $t < t(\omega'^n), t(\omega''^n)$. In the former case, $[\omega'^n] = [\omega''^n]$. In the latter case, $[\omega'^n]|_t = \omega'^n|_t = \omega''|_t = [\omega''^n]|_t$. $\square$

*Proof of Proposition 1.* Define $X^n_t(\omega) := P^n(\omega^n \notin \Omega^{*n} \mid \omega^n|_t)$. Extend the sequence by one date by defining $X^n_{T+1} = X^n_T$. It is common knowledge $X^n$ is a nonnegative martingale with expected value $X^n_0 = P^n(\omega^n \notin \Omega^{*n}) \leq \frac{c}{N-1} \cdot \frac{\varepsilon}{\overline{\overline{R}}}$.

Let $\tau^n$ denote the stopping time when $X^n_t$ first weakly exceeds $\frac{\varepsilon}{\overline{\overline{R}}}$. If $X^n_t$ never weakly exceeds $\frac{\varepsilon}{\overline{\overline{R}}}$, then set $\tau^n = T+1$. Let $E^n \subset \Omega$ denote the event $\tau^n \leq T$. By Doob's optional stopping theorem, we have

$$\frac{c}{N-1} \cdot \frac{\varepsilon}{\overline{\overline{R}}} \geq X^n_0 = \mathbf{E}X^n_{\tau^n} = \mathbf{E}X^n_{\tau^n}1_{\tau^n \leq T} + \mathbf{E}X^n_{\tau^n}1_{\tau^n = T+1} \geq \mathbf{E}X^n_{\tau^n}1_{\tau^n \leq T}.$$

Since $\mathbf{E}X^n_{\tau^n}1_{\tau^n \leq T} \geq \frac{\varepsilon}{\overline{\overline{R}}} \cdot P(E^n)$, it is common knowledge $P(E^n) \leq \frac{c}{N-1}$.

Fix an $n$ and let $\omega \notin \cup_{m \neq n} E^m$. Then $\forall t, m \neq n$ $P^m(\omega^m \notin \Omega^{*m} \mid \omega^m|_t) < \frac{\varepsilon}{\overline{\overline{R}}} \leq \frac{\varepsilon}{R^m_t(\omega^m|_t)}$. Let $\sigma \in \Sigma(id)$. Then, by definition, $\sigma^{-n}(\omega^{-n}) = \omega^{-n}$. Thus, by de Morgan's Law, $P(\sigma^{-n}(\omega^{-n}) \neq \omega^{-n}) \leq P(\cup_{m \neq n} E^m) \leq c \leq \varepsilon$. $\square$

*Proof of Corollary 1.* Let $\omega^n \notin \Omega^{*n}$. The proof of Proposition 1 implies that $X^n_T(\omega) = 1$ and $\omega \in E^n$. Thus, $\omega^n \notin E^n \Rightarrow \omega \in \Omega^{*n}$. Let $\sigma \in \Sigma(id)$ and $\omega \notin \cup_n E^n$. Then $\sigma(\omega) = \omega$ and $\omega \in \Omega^*$. Thus, $M^* \circ [\sigma(\omega)] = M^*(\omega) \in DO(\omega)$. By de Morgan's Law, $P(\cup_n E^n) \leq \frac{Nc}{N-1}$. $\square$

## 4.1 Proof of Theorem 1

I specialize to the case where $\omega^{avg} = \omega^{avg,n}$ for all $n$. The proof is easily adapted to the general case.

### 4.1.1 Constructing the Linked VCG Mechanisms

Given $N$ and any direct mechanism of the $N$-model, we can set $\overline{R}(N) := N^2 q f'(0) \vee \varepsilon$.

**Lemma 2.** *There exists a family of positive reals, $\{\varepsilon(N), x(N)\}_{N\geq 2}$ such that*

$$\begin{aligned} \varepsilon(N) &\leq \varepsilon & \forall N, \\ Ng(I(x(N))N) &\leq \frac{\frac{\varepsilon(N)}{N}\varepsilon}{(N-1)\overline{R}(N)} & \forall N, \\ \lim_{N\to\infty} \varepsilon(N) &= \lim_{N\to\infty} x(N) = 0. \end{aligned}$$

*Proof.* Given integer $k > 0$, since $\lim_{z\to\infty} z^5 g(z) = 0$, we have

$$\lim_{N\to\infty} (N-1)N^2\overline{R}(N) g\left(I\left(\frac{1}{k}\right)N\right) = 0.$$

Thus, there exists an $N_k$ such that for all $N \geq N_k$,

$$(N-1)N^2\overline{R}(N) g\left(I\left(\frac{1}{k}\right)N\right) < \frac{\varepsilon}{k}\cdot\varepsilon.$$

Obviously, the sequence of $N_k$ can be chosen to be strictly increasing.

Since $\lim_{x\to\infty} I(x) = \infty$, there exists an $x_0$ such that

$$N_1 g(I(x_0)2) \leq \frac{\frac{\varepsilon}{N_1}\varepsilon}{(N_1-1)\overline{R}(N_1)}.$$

For $N < N_1$, define $\varepsilon(N) = \varepsilon$ and $x(N) = x_0$. For all integers $k > 0$, and $N \in \{N_k, N_k+1, \ldots N_{k+1}-1\}$, define $\varepsilon(N) = \frac{\varepsilon}{k}$ and $x(N) = \frac{1}{k}$. The lemma is proved. □

Fix a family, $\{\varepsilon(N), x(N)\}_{N\geq 2}$, as in Lemma 2, and define, for each $N$,

$$\Omega^{*n}(N) = \left\{\omega^n \in \Omega^n(N) \;\middle|\; \left|\frac{\sum_{s\neq t}\omega_s^n}{N-1} - \omega^{avg}\right| \leq x(N) \;\; \forall t \leq N\right\}$$

for all $n \leq N$.

This yields a family of $\{\Omega^*(N)\}_{N\geq 2}$, and, consequently, a family of linked VCG mechanisms $\{(\Omega^*(N), V^*(N))\}_{N\geq 2}$.

Proposition 1 now implies that $id$ is an $\varepsilon$-ex-post equilibrium in each of these linked VCG mechanisms and it is common knowledge that

$$P(N)\left(\omega \in \Omega^*(N),\ \sigma(\omega) = \omega\right) > 1 - \frac{\varepsilon(N)}{N-1} \quad \forall \sigma \in \Sigma(N)(id).$$

This proves the first part of Theorem 1.

#### 4.1.2 Efficiency

**Lemma 3.** *There exists a family of positive reals, $\{y(N)\}_{N\geq 2}$ such that*

$$\lim_{N\to\infty}(N-1)N^2G(I(y(N))N) = 0,$$
$$\lim_{N\to\infty} y(N) = 0.$$

*Proof.* The proof is similar to that of Lemma 2. For each integer $k$, there exists an $N_k$ such that for all $N \geq N_k$,

$$(N-1)N^2G\left(I\left(\frac{1}{k}\right)N\right) \leq \frac{1}{k}.$$

$N_k$ can be chosen to be strictly increasing in $k$. Fix an arbitrary $y_0 > 0$. Then define $y(N) = y_0$ for all $N < N_1$, and $y(N) = \frac{1}{k}$ for all $N \in \{N_k, N_k+1, \ldots N_{k+1}-1\}$. □

Fix a family, $\{y(N)\}_{N\geq 2}$ as in Lemma 3, and define

$$\Omega^{**}(N) = \left\{\omega \;\middle|\; \left|\frac{\sum_{m\neq n}\omega_t^m}{N-1} - \omega^{avg}\right| \leq y(N)\ \forall t, n \leq N,\ \omega \in \Omega^*(N),\ \sigma(\omega) = \omega\right\}$$

It is common knowledge that $\lim_{N\to\infty} P(N)(\Omega^{**}(N)) \geq \lim_{N\to\infty} 1 - N^2G(I(y(N))N) - \frac{\varepsilon(N)}{N-1} = 1$.

**Lemma 4.** *The surpluses generated by $\{(\Omega^*(N), V^*(N))\}_{N\geq 2}$ satisfy the following bounds:*

$$\inf_{\omega\in\Omega^{**}(N)} S(\boldsymbol{A}(\omega), \omega) \geq N^2(\omega^{avg} - y(N))f\left(\frac{q}{\omega^{avg} - y(N)}\right),$$
$$\sup_{\omega\in\Omega(N)} S(\boldsymbol{A}(\omega), \omega) \leq N^2 q f'(0).$$

*Proof.* $\omega \in \Omega^{**}(N)$ implies $\left|\frac{\sum_{m\neq n}\omega_t^m}{N-1} - \omega^{avg}\right| \leq y(N)$ for all $n, t \leq N$, which then implies $\left|\frac{\sum_n \omega_t^n}{N} - \omega^{avg}\right| \leq y(N)$ for all $t \leq N$. Thus, the lowest possible surplus is generated when $\sum_n \omega_t^n = N(\omega^{avg} - y(N))$ for all $t \leq N$. In this case, the surplus generated is $N^2(\omega^{avg} - y(N)) f\left(\frac{q}{\omega^{avg} - y(N)}\right)$. On the other hand, the surplus generated is always weakly less than what is generated if $\omega_t^n = \infty$ for all $n, t \leq N$. In this case, the surplus generated is $N^2 q f'(0)$. □

Since $\sigma(\omega) = \omega$ for all $\sigma \in \Sigma(N)(id)$ and whenever $\omega \in \Omega^{**}$, the second part of Lemma 4 implies

$$\mathbf{E}_{P(N)} S(\boldsymbol{A}|_{\Omega^*(N)} \circ [\sigma(\omega)], \omega) \geq \mathbf{E}_{P(N)} S(\boldsymbol{A}(\omega), \omega) - (1 - P(\Omega^{**}))N^2 q f'(0).$$

And now, the first part of Lemma 4 implies

$$\lim_{N\to\infty} \inf_{\sigma \in \Sigma(N)(id)} \frac{\mathbf{E}_{P(N)} S(\boldsymbol{A}|_{\Omega^*(N)} \circ [\sigma(\omega)], \omega)}{\mathbf{E}_{P(N)} S(\boldsymbol{A}(\omega), \omega)} \geq \lim_{N\to\infty} 1 - \frac{(1 - P(N)(\Omega^{**}(N)))q f'(0)}{P(N)(\Omega^{**}(N))(\omega^{avg} - y(N)) f\left(\frac{\bar{q}}{\omega^{avg} - y(N)}\right)} = 1.$$

This proves the second part of Theorem 1.

#### 4.1.3 Expected Cost

**Lemma 5.** *Let $\omega^n \in \Omega^{*n}(N)$. Then $\omega_t^n \leq \omega^{avg} + (2N-1)x(N)$ for all $t \leq N$.*

*Proof.* $\omega^n \in \Omega^{*n}(N)$ implies $\left|\frac{\sum_{s\neq t} \omega_s^n}{N-1} - \omega^{avg}\right| \leq x(N)$ for all $t \leq N$, which then implies $\left|\frac{\sum_s \omega_s^n}{N} - \omega^{avg}\right| \leq x(N)$ or, equivalently, $\left|\sum_{s\neq t} \omega_s^n + \omega_t^n - N\omega^{avg}\right| \leq Nx(N)$ for all $t \leq N$. Thus,

$$\begin{aligned} \omega_t^n \leq N\omega^{avg} - \sum_{s\neq t} \omega_s^n + Nx(N) &\leq N\omega^{avg} - (N-1)(\omega^{avg} - x(N)) + Nx(N) \\ &= \omega^{avg} + (2N-1)x(N). \end{aligned}$$

□

Given $\omega \in \Omega^{**}(N)$, and $\hat{\omega}^n \in \Omega^{*n}(N)$, we have, by the concavity of $f$,

$$\begin{aligned} V^n(\omega) - V^n(\omega^{-n}, \hat{\omega}^n) = \sum_{t=1}^{N} \Bigg[ &\sum_{m\neq n} \omega_t^m f\left(\frac{Nq}{\omega_t^n + \sum_{m\neq n} \omega_t^m}\right) \\ &- \sum_{m\neq n} \omega_t^m f\left(\frac{Nq}{\hat{\omega}_t^n + \sum_{m\neq n} \omega_t^m}\right)\Bigg] \end{aligned}$$

$$\leq \sum_{t=1}^{N} \left[ f'\left(\frac{Nq}{\hat{\omega}_t^n + \sum_{m\neq n}\omega_t^m}\right)\left(\sum_{m\neq n}\omega_t^m\right) \cdot \left(\frac{Nq}{\omega_t^n + \sum_{m\neq n}\omega_t^m} - \frac{Nq}{\hat{\omega}_t^n + \sum_{m\neq n}\omega_t^m}\right)\right]$$

$$= \sum_{t=1}^{N} \left[ f'\left(\frac{Nq}{\hat{\omega}_t^n + \sum_{m\neq n}\omega_t^m}\right)\left(\sum_{m\neq n}\omega_t^m\right) \cdot \left(\frac{(\hat{\omega}_t^n - \omega_t^n)Nq}{\left(\omega_t^n + \sum_{m\neq n}\omega_t^m\right)\left(\hat{\omega}_t^n + \sum_{m\neq n}\omega_t^m\right)}\right)\right]. \quad (3)$$

This expression then implies

**Lemma 6.** *Linked VCG transfers are asymptotically dominated by $N$ over $\Omega^{**}(N)$.*

*Formally, there exists a family of positive reals $\{a(N), b(N)\}_{N\geq 2}$ satisfying $a(N) < b(N)$ for all $N$, and $\lim_{N\to\infty} a(N) = \lim_{N\to\infty} b(N) < \infty$, such that for all agents $n \leq N$ and $\omega \in \Omega^{**}(N)$, we have*

$$V^{*n}(\omega) \leq N2x(N)a(N) + N(\omega^{avg} + x(N))(b(N) - a(N)).$$

*Proof.* Consider the quantity

$$f'\left(\frac{Nq}{\hat{\omega}_t^n + \sum_{m\neq n}\omega_t^m}\right)\left(\sum_{m\neq n}\omega_t^m\right) \cdot \left(\frac{Nq}{\left(\omega_t^n + \sum_{m\neq n}\omega_t^m\right)\left(\hat{\omega}_t^n + \sum_{m\neq n}\omega_t^m\right)}\right). \quad (4)$$

Given $\omega \in \Omega^{**}(N)$, and $\hat{\omega}^n \in \Omega^{*n}(N)$, let us bound from above and below this quantity.

Lemma 5 and the fact that $\omega \in \Omega^{**}(N)$ imply

$$f'\left(\frac{Nq}{\hat{\omega}_t^n + \sum_{m\neq n}\omega_t^m}\right) \in \left[f'\left(\frac{\frac{N}{N-1}q}{\omega^{avg} - y(N)}\right), f'\left(\frac{\frac{N}{N-1}q}{\frac{\omega^{avg}}{N-1} + \frac{2N-1}{N-1}x(N) + \omega^{avg} + y(N)}\right)\right].$$

Similarly,

$$\left(\sum_{m\neq n}\omega_t^m\right) \cdot \left(\frac{Nq}{\left(\omega_t^n + \sum_{m\neq n}\omega_t^m\right)\left(\hat{\omega}_t^n + \sum_{m\neq n}\omega_t^m\right)}\right) \in$$

$$\left[\frac{(\omega^{avg} - y(N))q}{(\omega^{avg} + y(N))\left(\frac{\omega^{avg}}{N-1} + \frac{2N-1}{N-1}x(N) + \omega^{avg} + y(N)\right)}, \frac{(\omega^{avg} + y(N))q}{(\omega^{avg} - y(N))(\omega^{avg} - y(N))}\right]$$

So, define

$$a(N) := f'\left(\frac{\frac{N}{N-1}q}{\omega^{avg} - y(N)}\right) \cdot \frac{(\omega^{avg} - y(N))q}{(\omega^{avg} + y(N))\left(\frac{\omega^{avg}}{N-1} + \frac{2N-1}{N-1}x(N) + \omega^{avg} + y(N)\right)},$$

$$b(N) := f'\left(\frac{\frac{N}{N-1}q}{\frac{\omega^{avg}}{N-1} + \frac{2N-1}{N-1}x(N) + \omega^{avg} + y(N)}\right) \cdot \frac{(\omega^{avg} + y(N))q}{(\omega^{avg} - y(N))(\omega^{avg} - y(N))}.$$

Clearly, $a(N) < b(N)$ for all $N$. Moreover

$$\lim_{N\to\infty} a(N) = \lim_{N\to\infty} b(N) = \frac{q}{\omega^{avg}} f'\left(\frac{q}{\omega^{avg}}\right).$$

And now, we can bound from above the right hand side of (3) by

$$\sum_{t=1}^{N} \left(\hat{\omega}_t^n b(N) - \omega_t^n a(N)\right) = \left(\sum_{t=1}^{N} \hat{\omega}_t^n - \sum_{t=1}^{N} \omega_t^n\right) a(N) + \sum_{t=1}^{N} \hat{\omega}_t^n \left(b(N) - a(N)\right)$$
$$\leq N2x(N)a(N) + N(\omega^{avg} + x(N))\left(b(N) - a(N)\right).$$

The result now follows from the observation that

$$V^{*n}(\omega) = \arg\max_{\hat{\omega}^n \in \Omega^{*n}} \left(V^n(\omega) - V^n(\omega^{-n}, \hat{\omega}^n)\right).$$

□

Applying Lemma 6, we have

$$\mathbf{E}_{P(N)} C(V^* \circ [\sigma(\omega)]) \leq N^2 2x(N)a(N) + N^2(\omega^{avg} + x(N))(b(N) - a(N))$$
$$+ \left[N^2 G(I(y(N))N) + \frac{\varepsilon(N)}{N-1}\right](N-1)N^2 q f'(0).$$

Also,

$$\mathbf{E}_{P(N)} S(\boldsymbol{A}|_{\Omega^*(N)} \circ [\sigma(\omega)], \omega) \geq P(N)(\Omega^{**}(N))N^2(\omega^{avg} - y(N)) f\left(\frac{q}{\omega^{avg} - y(N)}\right).$$

Putting everything together and the third and final part of Theorem 1 is proved:

$$\lim_{N\to\infty} \sup_{\sigma \in \Sigma(N)(id)} \frac{\mathbf{E}_{P(N)} C(V^* \circ [\sigma(\omega)])}{\mathbf{E}_{P(N)} S(\boldsymbol{A}|_{\Omega^*(N)} \circ [\sigma(\omega)], \omega)}$$
$$\leq \lim_{N\to\infty} \frac{2x(N)a(N) + (\omega^{avg} + x(N))(b(N) - a(N)) + \left[(N-1)N^2 G(I(y(N))N) + \varepsilon(N)\right] q f'(0)}{P(N)(\Omega^{**}(N))(\omega^{avg} - y(N)) f\left(\frac{q}{\omega^{avg} - y(N)}\right)}$$
$$= 0.$$

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